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LETTER TO THE EDITOR

Invasion percolation in an external field: dielectric breakdown in random media

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Abstract. We propose a model to describe electric discharge in inhomogeneous insulators by introducing an external Laplacian field in the invasion percolation model. The patterns generated have a fractal structure with a tunable fractal dimension.

The growth of fractal interfaces has recently become an intensively studied problem [1, 2]. In general, there are two important reasons for the appearance of such highly ramified structures: it is either the inhomogeneity of the medium or a non-local diffusion-type field which results in fractal growth. A combination of these two major approaches is expected to contribute to the understanding of growth processes in random media.

The problem of capillary dominated two-phase fluid flow in porous media has recently been investigated using the invasion percolation (IP) model [3-5]. In IP the random medium is represented by a regular lattice in which each bond of the lattice is assigned a random number, drawn from a uniform distribution on the unit interval [0, 1]. The random numbers specify the pressure at which a bond can be filled with the invading fluid. The advance of the invading fluid consists of a sequence of discrete steps in which, at each stage, that pore (bond) on the interface which has the highest threshold capillary pressure, i.e. the one with the largest assigned random number, is filled.

Invasion percolation is applicable to the description of the moving interface when a non-wetting fluid is replacing a wetting one in an inhomogeneous medium. In the case, however, when a less viscous fluid is injected into a porous medium filled with a more viscous fluid an additional instability appears and the invading fluid develops long, branching fingers [6-8]. This means that in addition to the inhomogeneity of the medium the pressure distribution in the replaced, more viscous fluid has to be also taken into account through the solution of the Laplace equation with moving boundaries. Similarly, when an electric discharge takes place in an inhomogeneous insulator, the potential u satisfies the same equation $\nabla^2 u = 0$. One possible way to simulate the behaviour of the moving boundary in a field satisfying the Laplace equation is provided by the dielectric breakdown model [9] and its variations [10, 11] which are also closely related to diffusion-limited aggregation [12]. Chen and Wilkinson [8] used an alternative approach to a problem related to ours by solving the Laplace equation numerically in order to describe the interface of a bubble growing in a network of channels with random radii.

In this letter we generalise the standard IP by introducing an external Laplacian field. We have investigated two different versions of our model which we propose as models of electrical discharge breakdown in dirty media.

Consider a hypercubic lattice with a charge source at the origin. The bonds are insulating and carry a breakdown coefficient B which is randomly distributed in the unit interval $[0, 1]$. The potential satisfies the Laplace equation $\nabla^2 u = 0$ where $u = 0$ on the conducting discharge pattern after the breakdown has taken place and $u = 1$ on a circle with a fixed large radius. We have studied the growth of the discharge pattern under two conditions. In model I, in each time step a single new bond which is the bond with the largest value of Bu^β on the interface breaks down, where β is an adjustable constant. In model II, all bonds on the interface have the chance of breaking down, with the probability $Bu^\beta / P_{\max}$ where $P_{\max}$ is the largest value of Bu^β. Thus bonds with the maximum value of Bu^β, i.e. $P_{\max}$, always break down and those having smaller values of Bu^β break with smaller probability. In both models, after each growth step, the potential u is 'relaxed' by replacing u on each lattice site by the average of the potential on the four nearest-neighbour lattice sites.

Initially we performed simulations of model I on a square lattice placing the growth site at the centre and a circle of unit potential at a distance $R_{\max}$, where $R_{\max}$ was always chosen to be much greater than the extent of the growing pattern. We observed strong anisotropy in the growth patterns such that only a single almost one-dimensional branch grows out from the seed. Although several branches develop along the way, only one main trunk keeps growing to infinity. For this reason we carried out most of our simulations of model I on a strip geometry where the initial growth site was placed at the centre of one side of a rectangular box and the other pole was the plate of the opposite side kept at constant potential of unity. We employed periodic boundary conditions for the transverse direction.

For the wide range of values of β (from $\frac{1}{2}$ to $\frac{1}{20}$) the discharge pattern appears qualitatively similar. Figure 1 shows a pattern of 800 steps on an 80×200 grid with $\beta = \frac{1}{4}$. The pattern appears to have branches and wiggles at all length scales. We have analysed the structure of the patterns to see if it has a scaling or fractal geometry [13]. In figure 2 we present log-log plots of N against L, where N is the average number of points belonging to the pattern which fall within a box of size L centred on a point on the pattern, averaged over the whole pattern. From figure 2 we see that $\log N$ appears to scale linearly with $\log L$ indicating that the patterns have a fractal structure with the fractal dimension D given by the relation

$$N \sim L^D. \tag{1}$$

The values of D for various β are shown in table 1.

In model II, we have studied the growth pattern on a square lattice with the initial growth site at the origin, because in this model we did not observe any anisotropy in the patterns. A typical pattern in the simulations of model II is shown in figure 3 for $\beta = 1$. The maximum span of this pattern is 180 sites across. The cluster contains 4630 particles. The most striking feature of this pattern is its lack of anisotropy due to the underlying square lattice. This is in contrast to the cross- and square-shaped patterns recently found in a related model [10] where growth occurred on a homogeneous, rather than a random, square lattice. It appears that, at least up to the cluster sizes studied here, the underlying lattice does not affect the shape of the cluster. However,

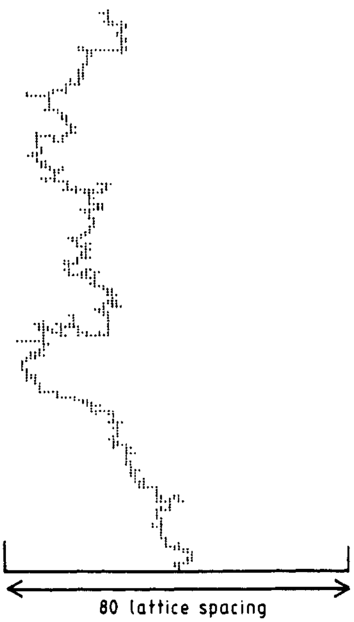

Figure 1. A pattern generated after 800 growth steps using model I with $\beta = \frac{1}{4}$ on a strip geometry with periodic boundary conditions in the transverse direction.

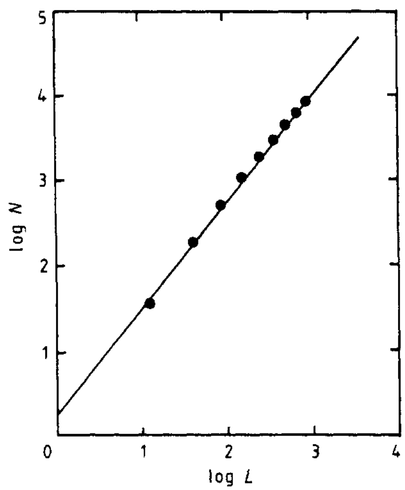

Figure 2. Log-log plot of N against L for model I, where N is the number of sites belonging to the pattern which fall within a box of size L centred on a point on the pattern, averaged over the whole pattern. The slope of the straight line gives $D = 1.25$ for $\beta = \frac{1}{4}$.

Table 1. Values of the fractal dimension D for various β in model I.

β	D
$\frac{1}{6}$	1.37
$\frac{1}{4}$	1.25
$\frac{1}{3}$	1.16

Figure 3. A cluster of 4630 sites generated using model II. In contrast to the patterns in a homogeneous medium (see reference [10]), the patterns generated in a random medium do not have an anisotropic structure for the same size clusters.

it is possible that in analogy with diffusion-limited aggregates, very large clusters eventually take on the fourfold symmetry of the square lattice [14-16].

In order to investigate the fractal properties of these clusters we have determined the dependence of the cluster size N on the radius of gyration R and the average number of particles $\bar{N}$ within a box of size L centred on the seed particle. The results, averaged over ten different growths up to $N \sim 5000$, are shown in figure 4. From the

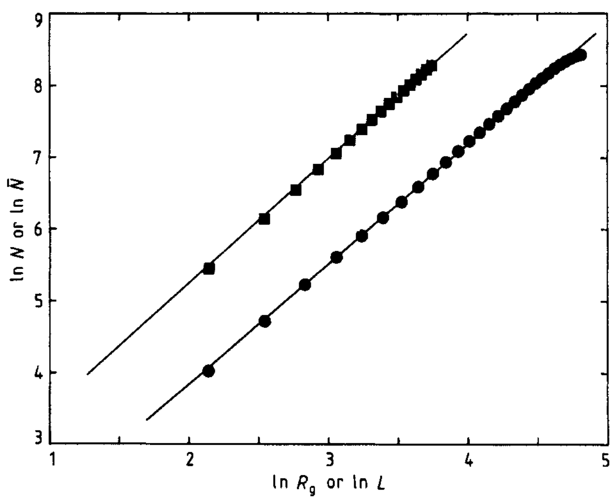

Figure 4. Log-log plot of the cluster size (number of sites) against the length of the cluster. The squares are the data for the number of sites in a cluster with a radius of gyration R_g. The full circles are for the average number of sites falling within a box of size L centred on a point in the cluster. The slope of the straight lines gives $D \sim 1.69$.

slopes of the straight lines in figure 4, the fractal dimensions D_{RG} and D_B determined from the relations $N \sim R^{D_{RG}}$ and $\bar{N} \sim L^{D_B}$ are found to be

$$D_{RG} = 1.70 \pm 0.05$$

$$D_B = 1.68 \pm 0.05.$$

These results agree with the estimates of D for DLA on a square lattice [15] and with the data for the same model in a homogeneous medium [10]. Thus, the presence of the random medium appears to change the shape of the clusters, but not their fractal dimension.

In conclusion, we studied the problem of electrical discharge patterns [17] in dirty media by solving the Laplace equation for the field and applying invasion percolation as a rule for breaking down a bond in the disordered lattice representing the random medium. In contrast to previous work on dielectric breakdown in homogeneous media the growth occurs deterministically in our models. In the first model only one point of the discharge pattern grows with time. This leads to an extremely stringy, almost linear, branch with a fractal dimension of about 1.2. In the second model, all bonds on the interface can break with varying probabilities that depend directly on the local electric field and the random breakdown factor. In this case the fractal dimension and the shape of the clusters are similar to those found for the dielectric breakdown model and diffusion-limited aggregate. However, up to the cluster sizes of about 5000, we did not observe any anisotropy with respect to the lattice structure.

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